THE GEOMETRY OF HRUSHOVSKI CONSTRUCTIONS, II. THE STRONGLY MINIMAL CASE.

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Abstract. We investigate the isomorphism types of combinatorial geometries arising from Hrushovski's flat strongly minimal structures and answer some questions from Hrushovski's original paper.

1. Introduction

In this paper, we investigate the isomorphism types of combinatorial geometries arising from Hrushovski's flat strongly minimal structures and answer some questions from Hrushovski's original paper [5]. It is a sequel to [2], but can be read independently of it. In order to describe the main results it will be convenient to summarise some of the results from the previous paper.

Suppose L is a relational language with, for convenience, all relation symbols of arity at least 3 and at most one relation symbol of each arity. Denote by $k(L)$ the maximum of the arities of the relation symbols in L (allowing $k(L)$ to be ∞ if this is unbounded). The basic Hrushovski construction defines the *predimension* of a finite L-structure to be its size minus the number of basic relations on the structure. The class $\mathcal{C}_0(L)$ consists of the finite L-structures in which this is non-negative on all substructures. There is then an associated notion of *dimension* d and the notion of *self-sufficiency* (denoted by $\leq$) of a substructure. All of this is reviewed in detail in Section 2 below. The class $(\mathcal{C}_0, \leq)$ has an associated *generic structure* $\mathcal{M}_0(L)$ which also carries a dimension function d giving it the structure of an infinite-dimensional pregeometry. The associated (combinatorial) geometry is denoted by $G(\mathcal{M}_0(L))$.

In [2] we showed that:

(1) The collection of finite subgeometries of $G(\mathcal{M}_0(L))$ does not depend on L (Theorem 3.8 of [2]).
(2) For languages L, L', the geometries $G(\mathcal{M}_0(L))$ and $G(\mathcal{M}_0(L'))$ are isomorphic iff the maximum arities $k(L)$ and $k(L')$ are equal.

(Theorem 3.1 of [2] for $\Leftarrow$ and see also Section 4.2 here; Theorem 4.3 of [2] gives $\Rightarrow$.)

(3) The localization of $G(\mathcal{M}_0(L))$ over any finite set is isomorphic to $G(\mathcal{M}_0(L))$ (Theorem 5.5 of [2]).

For the strongly minimal set construction of [5], one takes a certain function μ (see section 2 here) and considers a subclass $\mathcal{C}_\mu(L)$ of $\mathcal{C}_0(L)$. For appropriate μ there is a generic structure $\mathcal{M}_\mu(L)$ for the class $(\mathcal{C}_\mu(L), \leq)$ which is strongly minimal. The dimension function given by the predimension is the same as the dimension in the strongly minimal set and we are interested in the geometry of this. Our main result here is that this process of 'collapse' is irrelevant to the geometry: under rather general conditions on μ we prove:

(4) The geometry $G(\mathcal{M}_\mu(L))$ of the strongly minimal set is isomorphic to the geometry $G(\mathcal{M}_0(L))$ (Theorem 3.1).

Sections 5.1 and 5.2 of Hrushovski's paper [5] give variations on the construction which produce strongly minimal sets with geometries different from the $G(\mathcal{M}_0(L))$. However, we show, answering a question from [5] (see also Section 3 of [4]):

(5) the geometries of the strongly minimal sets in Sections 5.1 and 5.2 of [5] have localizations (over a finite set) which are isomorphic to one of the geometries $G(\mathcal{M}_0(L))$ (for appropriate L) (see Section 4.1 here).

The first version of the result in (4) was proved by the second Author in his thesis [3]: this was for the case where L has a single 3-ary relation symbol (as in the original paper [5]). The somewhat different method of proof used in Sections 3 and 4 here was found later. It has the advantage of being simpler and more readily adaptable to generalization and proving the result in (5), however, the class of μ-functions to which it is applicable is slightly more restricted than the result from [3]: Theorem 6.2.1 of [3] assumes only that $\mu \geq 1$.

In summary, for each $k = 3, 4, \ldots, \infty$ we have a countably-infinite dimensional geometry $\mathcal{G}_k$ isomorphic to $G(\mathcal{M}_0(L))$ where L has maximum arity k, and these are pairwise non-isomorphic. The geometry of each of the new (countable, saturated) strongly minimal sets in [5] has a localization isomorphic to one of these $\mathcal{G}_k$. Thus, whilst there is some diversity amongst the strongly minimal structures which can be produced by these constructions, the range of geometries which can be produced appears to be rather limited. It would therefore be very interesting to have a characterization of the geometries $\mathcal{G}_k$ in terms of a 'geometric' condition (such as flatness, as in 4.2 of [5], for example) and a condition on the automorphism group (such as homogeneity, but possibly with a stronger assumption).

Acknowledgement: Some of the results of this paper were produced whilst the second Author was supported as an Early Stage Researcher by the Marie Curie Research Training Network MODNET, funded by grant MRTN-CT-2004-512234 MODNET from the CEC.

2. Hrushovski constructions

We give a brief description of Hrushovski's constructions from [5]. Other presentations can be found in [7] and [1]. The book [6] of Pillay contains all necessary background material on pregeometries and model theory. The notation, terminology and level of generality is mostly consistent with that used in [2].

2.1. **Predimension and pregeometries.** Let L be a relational language consisting of relation symbols $(R_i : i \in I)$ with R_i of arity $n_i \geq 3$. We suppose there are only finitely many relations of each arity here.

We work with L-structures A where each each R_i is symmetric: so we regard the interpretation R_i^A of R_i in A as a set of n_i-sets. (By modifying the language, the arguments we give below can be adapted to deal with the case of n_i-tuples of not-necessarily-distinct elements: see Section 4.3 here.)

For finite A we let the predimension of A be $\delta(A) = |A| - \sum_{i\in I} |{R_i}^A|$ (of course this depends on L but this will be clear from the context).

We let $\mathcal{C}_0(L)$ be the set of finite L-structures A such that $\delta(A') \geq 0$ for all $A' \subseteq A$.

Suppose $A \subseteq B \in \mathcal{C}_0(L)$. We write $A \leq B$ and say that A is *self-sufficient* in B if for all B' with $A \subseteq B' \subseteq B$ we have $\delta(A) \leq \delta(B')$. We will assume that the reader is familiar with the basic properties (such as transitivity) of this notion.

Let $\bar{\mathcal{C}}_0(L)$ be the class of L-structures all of whose finite substructures lie in $\mathcal{C}_0(L)$. We can extend the notion of self-sufficiency to this class in a natural way.

Note that if $A \subseteq B \in \bar{\mathcal{C}}_0(L)$ is finite then there is a finite A' with $A \subseteq A' \subseteq B$ and $\delta(A')$ as small as possible. In this case $A' \leq B$ and it can be shown that there is a smallest finite set $C \leq B$ with $A \subseteq C$. We define the *dimension* $d_B(A)$ of A (in B) to be the minimum value of $\delta(A')$ for all finite subsets A' of B which contain A.

We define the *d-closure* of A in B to be:

$$\mathrm{cl}_B(A) = \{c \in B : d_B(Ac) = d_B(A)\}$$

where, as usual, Ac is shorthand for $A \cup \{c\}$.

These notions can be relativized: if $A, C \subseteq B \in \mathcal{C}_0(L)$ define $\delta(A/C)$ to be $\delta(A \cup C) - \delta(C)$ and $d_B(A/C) = d_B(A \cup C) - d_B(C)$.

We can coherently extend the definition of d-closure to infinite subsets A of B by saying that the d-closure of A is the union of the d-closures of finite subsets of A. It can be shown that (B, cl_B) is a pregeometry and the dimension function (as cardinality of a basis) equals d_B on finite subsets of B. We use the notation $PG(B)$ instead of (B, cl_B), and denote by $G(B)$ the associated (combinatorial) geometry: so the elements of $G(B)$ are the sets $\mathrm{cl}_B(x) \setminus \mathrm{cl}_B(\emptyset)$ for $x \in B \setminus \mathrm{cl}_B(\emptyset)$ and the closure on $G(B)$ is that induced by cl_B. Note that if $A \leq B \in \bar{\mathcal{C}}_0(L)$ then for $X \subseteq A$ we have $d_A(X) = d_B(X)$. Thus G can be regarded as a functor from $(\bar{\mathcal{C}}_0(L), \leq)$ to the class of geometries (with embeddings of geometries as morphisms).

If $Y \subseteq B \in \bar{\mathcal{C}}_0(L)$ then the *localization* of $PG(B)$ over Y is the pregeometry with closure operation $\mathrm{cl}_B^Y(Z) = \mathrm{cl}_B(Y \cup Z)$. The corresponding geometry is denoted by $G_Y(B)$. Note that the dimension function here is given by the relative dimension $d_B(./Y)$.

It will be convenient to fix a first order language for the class of pregeometries. A reasonable choice for this is the language $LPI = \{I_n : n \geq 1\}$ where each I_n is an n-ary relational symbol. A pregeometry (P, cl) will be seen as a structure in this language by taking I_n^P to be the set of independent n-tuples in P. Notice that we can recover a pregeometry just by knowing its finite independent sets. Note also that the isomorphism type of a pregeometry is determined by the isomorphism type of its associated geometry and the size of the equivalence classes of interdependence. In the case where these are all countably infinite, it therefore makes no difference whether we consider the geometry or the pregeometry.

2.2. **Self-sufficient amalgamation classes.** If $B_1, B_2 \in \bar{\mathcal{C}}_0(L)$ have a common substructure A then the *free amalgam* E of B_1 and B_2 over A consists of the disjoint union of B_1 and B_2 over A and $R_i^E = R_i^{B_1} \cup R_i^{B_2}$ for each $i \in I$. It is well known that if $A \leq B_1$ then $B_2 \leq E$, so $E \in \bar{\mathcal{C}}_0(L)$, and $(\mathcal{C}_0, \leq)$ is an *amalgamation class*. It can also be shown that if A is d-closed in B_1 then B_2 is d-closed in E.

Suppose $Y \leq Z \in \mathcal{C}_0(L)$ and $Y \neq Z$. Following [5], we say that this is an *algebraic extension* if $\delta(Y) = \delta(Z)$. It is a simply algebraic extension if also $\delta(Z') > \delta(Y)$ whenever $Y \subset Z' \subset Z$. It is a minimally simply algebraic (msa) extension if additionally $Y' \subseteq Y' \cup (Z \setminus Y)$ is not simply algebraic whenever $Y' \subset Y$.

The following is trivial, but crucial for us:

Lemma 2.1. *Suppose $Y \leq Z$ is a msa extension. Then for every $y \in Y$ there is some $w \in \bigcup_{i \in I} R_i^Z$ and $z \in Z \setminus Y$ such that $y, z \in w$. Moreover, if $Z \setminus Y$ is not a singleton and $z \in Z \setminus Y$, then there are at least two elements of $\bigcup_{i \in I} R_i^Z$ which contain z.*

Proof. Suppose this does not hold for some $y \in Y$. Let $Y' = Y \setminus \{y\}$. Then for every $U \subseteq Z \setminus Y$ we have $\delta(Y' \cup U) - \delta(Y') = \delta(Y \cup U) - \delta(Y)$. So $Y' \subseteq Y' \cup (Z \setminus Y)$ is simply algebraic: contradiction. Similarly, for the 'moreover' part, if z is in at most one relation in $\bigcup_{i \in I} R_i^Z$, then $\delta(Z \setminus \{z\}) \leq \delta(Z)$, which contradicts the simple algebraicity. □

We let μ be a function from the set of isomorphism types of minimally simply algebraic extensions in $\mathcal{C}_0(L)$ to the natural numbers. The subclass $\mathcal{C}_\mu(L)$ consists of structures in $\mathcal{C}_0(L)$ which, for each msa $Y \leq Z$ in $\mathcal{C}_0(L)$, omit the atomic type consisting of $\mu(Y, Z) + 1$ disjoint copies of Z over Y.

We will work with μ where the following holds:

Assumption 2.2 (Assumed Amalgamation Lemma).

(i) *If $A \leq B_1, B_2 \in \mathcal{C}_\mu(L)$ and the free amalgam of B_1 and B_2 over A is not in $\mathcal{C}_\mu(L)$, then there exists $Y \subseteq A$ and minimally simply algebraic extensions $Y \leq Z_i \in B_i$ (for $i = 1, 2$) which are isomorphic over Y and $Z_i \setminus Y \subseteq B_i \setminus A$.*

(ii) *The class $(\mathcal{C}_\mu(L), \leq)$ is an amalgamation class (see below).*

Note that (ii) here follows from (i) (cf. the proof of Lemma 4 in [5]), and by Section 2 of [5], (i) holds if $\mu(Y, Z) \geq \delta(Y)$ for all msa $Y \leq Z$ in $\mathcal{C}_0(L)$.

2.3. **Generic structures and their geometries.** Suppose $\mathcal{A}$ is a subclass of $\mathcal{C}_0(L)$ such that $(\mathcal{A}, \leq)$ is an amalgamation class: meaning that if $B \in \mathcal{A}$ and $A \leq B$ then $A \in \mathcal{A}$, and if $A \leq B_1, B_2 \in \mathcal{A}$ then there is $C \in \mathcal{A}$ and embeddings $f_i : B_i \to C$ with $f_i(B_i) \leq C$ and $f_1|A = f_2|A$. Then there is a countable structure $\mathcal{M} \in \bar{\mathcal{C}}_0(L)$ satisfying the following conditions:

(G1) $\mathcal{M}$ is the union of a chain $A_0 \leq A_1 \leq A_2 \leq \cdots$ of structures in $\mathcal{A}$.

(G2) (extension property) If $A \leq \mathcal{M}$ and $A \leq B \in \mathcal{A}$ then there exists an embedding $g : B \to \mathcal{M}$ such that $g(B) \leq \mathcal{M}$ and $g(a) = a$ for all $a \in A$.

We refer to $\mathcal{M}$ as the *generic structure* of the amalgamation class $(\mathcal{A}, \leq)$: it is determined up to isomorphism by the properties G1 and G2 (and G1 is automatic for countable structures in $\bar{\mathcal{C}}_0(L)$). Of course, Hrushovski's strongly minimal sets are the generic structures $\mathcal{M}_\mu(L)$

for the amalgamation classes $(\mathcal{C}_\mu(L), \leq)$. We will compare the geometries of these with that of the generic structure $\mathcal{M}_0(L)$ for the amalgamation class $(\mathcal{C}_0(L), \leq)$.

Suppose $(\mathcal{A}, \leq)$ and $(\mathcal{A}', \leq)$ are amalgamation classes, as above. We refer to the following as the Isomorhism Extension Property, and denote it by $\mathcal{A} \rightsquigarrow \mathcal{A}'$.

(*) Suppose $A \in \mathcal{A}$, $A' \in \mathcal{A}'$ and $f : G(A) \to G(A')$ is an isomorphism of geometries, and $A \leq B \in \mathcal{A}$. Then there is $B' \in \mathcal{A}'$ with $A' \leq B'$ and an isomorphism $f' : G(B) \to G(B')$ which extends f.

Lemma 2.3. *Suppose $(\mathcal{A}, \leq)$ and $(\mathcal{A}', \leq)$ are amalgamation classes with generic structures $\mathcal{M}$, $\mathcal{M}'$ respectively. Suppose that both extension properties $\mathcal{A} \rightsquigarrow \mathcal{A}'$ and $\mathcal{A}' \rightsquigarrow \mathcal{A}$ hold. Then the geometries $G(\mathcal{M})$ and $G(\mathcal{M}')$ are isomorphic.*

Proof. We have already remarked that if $A \leq \mathcal{M}$, then the dimension of a subset of A is the same whether computed in A or in $\mathcal{M}$. Thus $G(A)$ is naturally a substructure of $G(\mathcal{M})$. We claim that the set $\mathcal{S}$ of geometry-isomorphisms

$$f : G(A) \to G(A')$$

where $A \leq \mathcal{M}, A' \leq \mathcal{M}'$ are finite is a back-and-forth system between $G(\mathcal{M})$ and $G(\mathcal{M}')$. Indeed (for the 'forth'), given such an $f : G(A) \to G(A')$ and $A \leq B \leq \mathcal{M}$, there is $A' \leq B' \in \mathcal{A}'$ and an isomorphism $f' : G(B) \to G(B')$ extending f, by our assumption $\mathcal{A} \rightsquigarrow \mathcal{A}'$. The extension property G2 in $\mathcal{M}'$ means that we can take $B' \leq \mathcal{M}'$, as required. Similarly we obtain the 'back' part from $\mathcal{A}' \rightsquigarrow \mathcal{A}$ and G2 in $\mathcal{M}$. It follows that $G(\mathcal{M})$ and $G(\mathcal{M}')$ are isomorphic. $\square$

3. Isomorphism of the strongly minimal set geometries

Throughout, $(\mathcal{C}_0(L), \leq)$ and $(\mathcal{C}_\mu(L), \leq)$ are the amalgamation classes from the previous section. Note that $(\mathcal{C}_0(L), \leq)$ is an amalgamation class and we are *assuming* that the amalgamation lemma 2.2 holds for $\mathcal{C}_\mu(L)$. We denote the generic structures by $\mathcal{M}_0(L)$ and $\mathcal{M}_\mu(L)$ respectively: so the latter is Hrushovski's strongly minimal set $D(L, \mu)$. The geometries are denoted by $G(\mathcal{M}_0(L))$ etc. Our main result is:

Theorem 3.1. *Suppose 2.2 holds and $\mu(Y, Z) \geq 2$ for all msa $Y \leq Z \in \mathcal{C}_0(L)$ with $\delta(Y) \geq 2$ and $\mu(Y, Z) \geq 1$ when $\delta(Y) = 1$. Then $G(\mathcal{M}_\mu(L))$ and $G(\mathcal{M}_0(L))$ are isomorphic geometries.*

Proof. We need to verify that the isomorphism extension property of Lemma 2.3 holds in both directions. The main part will be to show that $\mathcal{C}_0(L) \rightsquigarrow \mathcal{C}_\mu(L)$.

So suppose we are given $A \leq B \in \mathcal{C}_0(L)$ and $A' \in \mathcal{C}_\mu(L)$ with an isomorphism $f : G(A) \to G(A')$. We want to find $B' \in \mathcal{C}_\mu(L)$ with $A' \leq B'$ and an isomorphism $f' : G(B) \to G(B')$ extending f. The main point will be to ensure that each point of $B' \setminus (A' \cup \mathrm{cl}_{B'}(\emptyset))$ is involved in only a small number of relations, and this gives us control over the msa extensions in B'.

Let $A_0 = \mathrm{cl}_A(\emptyset)$ and let $A_1, \ldots, A_r$ be the d-dependence classes on $A \setminus A_0$: the latter are the points of $G(A)$. Similarly let $B_0 = \mathrm{cl}_B(\emptyset)$ and $B_1, \ldots, B_s$ the d-dependence classes on $B \setminus B_0$, with $A_i \subseteq B_i$ for $i = 1, \ldots, r$. List the relations on B which are not contained in A or some $B_0 \cup B_j$ as $\rho_1, \ldots, \rho_t$. So these are finite sets. Let $A'_0 = \mathrm{cl}_{A'}(\emptyset)$ and $A'_1, \ldots, A'_r$ be the classes of d-dependence on $A' \setminus A'_0$, labelled so that $f(A_i) = A'_i$. We construct $B'_0, B'_1, \ldots, B'_s$ with $A'_i \subseteq B'_i$ for $i = 0, \ldots, r$, and $B' = \bigcup_{i=0}^{s} B'_i$ in the steps below.

Terminology: If $u, v \in E \in \mathcal{C}_0(L)$, say that u, v are *adjacent in E* if there exists $w \in \bigcup_i R_i^E$ such that $u, v \in w$.

Step 1: Construction of $A'' = A' \cup B'_0 \in \mathcal{C}_\mu(L)$.
Take $A'_0 \leq V \in \mathcal{C}_\mu(L)$ with $\delta(V) = 0$ and $|V \setminus A'_0|$ sufficiently large. (For example, it is easy to show that $\mathcal{C}_\mu(L)$ contains arbitrarily large structures of δ-value 0; take V to be the disjoint union of A'_0 and one of these.) Let A'' be the free amalgam of A' and V over A'_0 and let B'_0 be the copy of V inside this. As A'_0 is d-closed in A' and $A'_0 \leq V$ it follows from 2.2 that $A'' \in \mathcal{C}_\mu(L)$, B'_0 is d-closed in A'' and $A' \leq A''$.

Step 2: Construction of $B'_0 \cup B'_i \in \mathcal{C}_\mu(L)$.
We do this so that B'_0 is d-closed in $B'_0 \cup B'_i$ and $\delta(B'_i \cup B'_0) = 1$. (As $\delta(B'_0 \cup A'_i) = 1$, it then follows that $B'_0 \cup A'_i \leq B'_0 \cup B'_i$ when $1 \leq i \leq r$.) Let m be sufficiently large. Choose some R_i: for example R_1 of arity $n \geq 3$.

Case 1: Suppose $i \leq r$. Pick $b_{i0} \in A'_i$ and let $s_{i1}, \ldots, s_{im}$ be disjoint $(n-2)$-subsets of $B'_0 \setminus A'_0$ (we adjust the choice of V in step 1 to accommodate this). Let $B'_i = A'_i \cup \{b_{i1}, \ldots, b_{im}\}$ and include as new R_1-relations on $B'_0 \cup B'_i$ the n-sets $\{b_{i0}, b_{ij}\} \cup s_{ij}$ for $1 \leq j \leq m$. We need to show that this has the required properties.

First, note that $B'_0 \cup A'_i \leq B'_0 \cup A'_i \cup \{b_{ij}\}$, so $B'_0 \cup A'_i \cup \{b_{ij}\} \in \mathcal{C}_0(L)$.

Suppose $Y \leq Z$ is a msa extension in $B'_0 \cup A'_i \cup \{b_{ij}\}$ not contained in $B'_0 \cup A'_i$. So $s_{ij} \cup \{b_{i0}, b_{ij}\} \subseteq Z$. If $b_{ij} \notin Y$ then $Y \leq Z \setminus \{b_{ij}\} < Z$

is algebraic, so $Z \setminus \{b_{ij}\} = Y$ and $Y = s_{ij} \cup \{b_{i0}\}$. As the elements of s_{ij} are non-adjacent to b_{i0} in $B'_0 \cup A'_i$, it follows that there is only one copy of Z over Y in $B'_0 \cup A'_i \cup \{b_{ij}\}$. If $b_{ij} \in Y$, then by Lemma 2.1 $s_{ij} \cup \{b_{i0}\} \not\subseteq Y$. But then there is at most one copy of Z over Y in $B'_0 \cup A'_i \cup \{b_{ij}\}$. Note that $\delta(Y) = \delta(Z) \geq \delta(Z \cap (A'_i \cup B'_0)) \geq 1$. So in both cases we meet the requirements for $B'_0 \cup A'_i \cup \{b_{ij}\} \in \mathcal{C}_\mu(L)$.

Now note that $B'_0 \cup B'_i$ is the free amalgam over $B'_0 \cup A'_i$ of the structures $B'_0 \cup A'_i \cup \{b_{ij}\}$ (for $j = 1, \ldots, m$). Each $B'_0 \cup A'_i \subseteq B'_0 \cup A'_i \cup \{b_{ij}\}$ is an algebraic extension and the only msa extension in this with base in $B'_0 \cup A'_i$ and which is not contained in $B'_0 \cup A'_i$ is $s_{ij} \cup \{b_{i0}\} \leq s_{ij} \cup \{b_{i0}, b_{ij}\}$. So the amalgamation lemma 2.2 implies that $B'_0 \cup A'_i \leq B'_0 \cup B'_i \in \mathcal{C}_\mu(L)$. It is clear that $\delta(B'_i/B'_0 \cup A'_i) = 0$ so $\delta(B'_0 \cup B'_i) = \delta(B'_0 \cup A'_i) = \delta(B'_0) + 1$.

Finally, note that as B'_0 is d-closed in $B'_0 \cup A'_i$ and $B'_0 \cup A'_i \leq B'_0 \cup B'_i$, the d-closure of B'_0 in $B'_0 \cup B'_i$ does not contain b_{i0}. It then follows easily that B'_0 is d-closed in $B'_0 \cup B'_i$.

Case 2: $i > r$. As in Case 1, let $s_{i1}, \ldots, s_{im}$ be $(n-2)$-subsets of $B'_0 \setminus A'_0$ with no relations on them. Let $B'_i = \{b_{i1}, \ldots, b_{im}\}$ and include as new R_1-relations on $B'_0 \cup B'_i$ the n-sets $\{b_{ij}, b_{i(j+1)}\} \cup s_{ij}$ for $1 \leq j \leq m-1$. In this version of the construction we take the s_{ij} to be s_i, independent of j.

It is clear that $\delta(B'_0 \cup B'_i) = 1$ and if $\emptyset \neq Y \subseteq B'_i$ then $\delta(B'_0 \cup Y) \geq 1$. So $B'_0 \leq B'_0 \cup B'_i$ and therefore $B'_0 \cup B'_i \in \mathcal{C}_0(L)$, and B'_0 is d-closed in $B'_0 \cup B'_i$. It remains to show that $B'_0 \cup B'_i \in \mathcal{C}_\mu(L)$, so suppose $Y \leq Z_1$ is a msa extension in $B'_0 \cup B'_i$. If $\delta(Y) = 0$ then $Z_1 \subseteq B'_0$ and the same is true of any copy of Z_1 over Y. Similarly, if $Y \subseteq B'_0$ then all copies of Z_1 over Y are contained in B'_0 as this is d-closed in $B'_0 \cup B'_i$. So we can assume that $\delta(Y) \geq 1$ and $b_{ij} \in Y$ for some j. By Lemma 2.1, we can assume that $s_i \cup \{b_{ij}, b_{i(j-1)}\} \subseteq Z_1$.

If $Z_1 \setminus Y$ is a singleton then there is at most one other copy Z_2 of Z_1 over Y, and in this case $Y = Z_1 \cap Z_2 = s_i \cup \{b_{ij}\}$. Note that $\delta(Y) \geq 2$ here, so $\mu(Y, Z_1) \geq 2$, by hypothesis.

Now suppose that $Z_1 \setminus Y$ has at least two elements. It will suffice to prove that there is no other copy Z_2 of Z_1 over Y in $B'_0 \cup B'_i$, so suppose there is such a Z_2. Take j maximal such that $b_{ij} \in Z_1 \cup Z_2$. By Lemma 2.1, b_{ij} is in at least two relation in $Z_1 \cup Z_2$; but b_{ij} is only in two relations in $B'_0 \cup B'_i$ and one of these also involves $b_{i(j+1)}$, so this is in $Z_1 \cup Z_2$. This contradicts the choice of j.

Step 3: Other relations on B'.
The relations on B' not contained in A' or some $B'_0 \cup B'_i$ are $\rho'_1, \ldots \rho'_t$. We can choose these to be subsets of $B' \setminus A''$ with $\rho'_i \cap \rho'_j = \emptyset$ if $i \neq j$,

and $\rho_i' \cap B_j' \neq \emptyset$ iff $\rho_i \cap B_j \neq \emptyset$ (for $j \geq 1$). Note that this is possible if m is sufficiently large. We make ρ_i' of the same type as ρ_i (that is, in the same R_j).

This completes the construction of B'. We now make a series of claims about it.

Claim 1: Let $U \subseteq \{1, \ldots, s\}$, and $Y = \bigcup_{i \in U}(B_i \cup B_0)$, $Y' = \bigcup_{i \in U}(B_i' \cup B_0')$. Then $Y \cap A$ is d-closed in A iff $Y' \cap A'$ is d-closed in A', and in this case we have $\delta(Y) = \delta(Y')$.

Let $U_0 = \{i \in U : i \leq r\}$ and $U_1 = \{i \in U : i > r\}$. Then $Y \cap A = \bigcup_{i \in U_0}(A_0 \cup A_i)$ and $Y' \cap A' = \bigcup_{i \in U_0}(A_0' \cup A_i')$. Because f is an isomorphism of geometries, one of these is d-closed (in A or A') iff the other is (remembering that a subset of a geometry is d-closed iff any set properly containing it has bigger dimension). So suppose this is the case. We compute that:

$$\delta(Y') = \delta(A'' \cap Y') + \delta(Y'/A'' \cap Y') = \delta(A'' \cap Y') + |U_1| - |J|$$

where $J = \{j : \rho_j' \subseteq Y'\}$. This follows from the fact that Y' consists of $|U_0|$ sets of δ-value 0 over $A'' \cap Y'$ and $|U_1|$ sets of δ-value 1 over $A'' \cap Y'$, and an extra $|J|$ relations ρ_j' between them. Moreover

$$\delta(A'' \cap Y') = \delta((A' \cap Y') \cup B_0') = \delta(A' \cap Y'),$$

using, for example, the construction of A'' as a free amalgam in step 1. Thus we have

$$\delta(Y') = \delta(A' \cap Y') + |U_1| - |J|.$$

Now, by construction (step 3) we have $\rho_j \subseteq Y$ iff $\rho_j' \subseteq Y'$. So an identical calculation shows that

$$\delta(Y) = \delta(A \cap Y) + |U_1| - |J|.$$

Now if one (and hence both) of $A' \cap Y'$, $A \cap Y$ is d-closed (in A', A respectively) then they have the same dimension, and therefore as they are d-closed they have the same δ-value. Thus, in this case $\delta(Y) = \delta(Y')$, as required. $\square_{Claim}$

Claim 2: $B' \in \mathcal{C}_0(L)$, B_0' and $B_0' \cup B_i'$ are d-closed in B' and $A'' \leq B'$. The map $f' : G(B) \to G(B')$ given by $f'(B_i) = B_i'$ is an isomorphism of geometries which extends f.

Note that (by construction step 2) if $B_0' \subseteq X \subseteq B'$ then $\delta(X) \geq \delta(\bigcup\{B_0' \cup B_i' : X \cap B_i' \neq \emptyset\})$. In particular, if $A'' \subseteq X$ then we can apply claim 1 to the latter to deduce that $\delta(X) \geq \delta(A') = \delta(A'')$, using the fact that $A \leq B$. So $A'' \leq B'$ and it follows that $\emptyset \leq B'$.

Suppose Y' is d-closed in B'. Then $B_0' \subseteq Y'$ (as $\delta(B_0') = 0$) and as above, Y' is of the form $\bigcup_{i \in U}(B_0' \cup B_i')$ for some U. Moreover $Y' \cap A'$ is

d-closed in A' and so we can apply claim 1. It follows from this that B_0' is d-closed in B' and the d-closed sets of dimension 1 are the $B_0' \cup B_i'$, by using the fact that the corresponding statements hold in B.

It remains to show that f' is an isomorphism of geometries. Let Y, Y' be as in claim 1. We need to show that Y is d-closed in B iff Y' is d-closed in B'. So suppose Y is d-closed in B. Then $Y \cap A$ is d-closed in A and so we can apply Claim 1 to get that $\delta(Y) = \delta(Y')$. Suppose for a contradiction that Y' is not d-closed in B'. Let Z' be its d-closure in B'. So $Z' = \bigcup_{i \in Q}(B_0' \cup B_i')$ for some Q with $U \subset Q \subseteq \{1, \ldots, s\}$ and $\delta(Z') \leq \delta(Y')$. Let $Z = \bigcup_{i \in Q}(B_0 \cup B_i)$. So $Y \subset Z$. Because Z' is d-closed in B' and therefore $Z' \cap A'$ is d-closed in A', we can apply Claim 1 to get that $\delta(Z) = \delta(Z')$. So we have

$$\delta(Z) = \delta(Z') \leq \delta(Y') = \delta(Y)$$

and this contradicts the fact that Y is d-closed and $Y \subset Z$. Thus Y' is d-closed in B'. The argument for the converse implication is the same. $\square_{Claim}$

Claim 3: $B' \in \mathcal{C}_\mu(L)$.

Suppose that $Y \leq Z$ is a minimally simply algebraic extension in B'. First suppose $\delta(Y) = \delta(Z) \leq 1$. Then $Y \subseteq B_0' \cup B_i'$ for some i and as $B_0' \cup B_i'$ is d-closed in B', any copies of Z over Y in B' are contained in $B_0' \cup B_i'$. So there are at most $\mu(Y, Z)$ of these as $B_0' \cup B_i' \in \mathcal{C}_\mu(L)$.

Now suppose that $\delta(Y) \geq 2$ and suppose for a contradiction that Z_i (for $i = 1, \ldots, \mu(Y, Z) + 1$) are disjoint copies in B' of Z over $Y \subseteq B'$ (meaning that the sets $Z_i \setminus Y$ are disjoint, of course).

If $y \in Y$ then y is in some relation in $R_k^Z \setminus R_k^Y$ (for some $k \in I$) by Lemma 2.1. Thus y is in at least three relations in $R_k^{B'}$ (one in each $R_k^{Z_i} \setminus R_k^Y$). By inspection of the construction one therefore sees that $y \in A''$ or $y = b_{ij}$ for some $i > r$. In the latter case, two of the (at most) three relations in B' which involve b_{ij} are $s_i \cup \{b_{i(j-1)}, b_{ij}\}$ and $s_i \cup \{b_{i(j+1)}, b_{ij}\}$. So we can assume that the first is a subset of Z_1 (and not a subset of Y) and the second is a subset of Z_2. But this implies that $s_i \subseteq Y$. However, there is no other relation which contains $\{b_{ij}\} \cup s_i$: contradicting the fact that Z_3 is a copy of Z_1 over Y.

Thus $Y \subseteq A''$. As $A'' \in \mathcal{C}_\mu(L)$ not all of the Z_i are subsets of A'', so we can assume that $Z_1 \not\subseteq A''$. As $A'' \leq B'$ we have $Y \subseteq A'' \cap Z_1 \leq Z_1$ so (by the simplicity of the extension) $Z_1 \cap A'' = Y$.

Note that Z_1 is in the d-closure of Y so we cannot have $Y \subseteq B_0'$. Let $y \in Y \setminus B_0'$. This is adjacent in Z_1 to some $z \in Z_1 \setminus Y$. So $y \in A'' \setminus B_0'$ is adjacent in B' to $z \in B' \setminus A''$. Inspection of the construction shows that $y = b_{i0}$ (for some $i \leq r$) and $z = b_{ij}$. Then the adjacency of y and

z in Z_1 forces $s_{ij} \subseteq Z_1$, and so $s_{ij} \subseteq Y$ (as $A'' \cap Z_1 = Y$). But then $Y \leq Y \cup \{b_{ij}\}$ is a simply algebraic extension in Z_1. As $Y \leq Z_1$ is a minimally simply algebraic extension, this implies $Y = \{b_{i0}\} \cup s_{ij}$ and $Z_1 = \{b_{i0}, b_{ij}\} \cup s_{ij}$. However, there is no other relation in B' which contains this Y (by construction), so we have a contradiction. $\square_{Claim}$

Claims 2 and 3 finish the proof of the isomorphism extension property $\mathcal{C}_0(L) \rightsquigarrow \mathcal{C}_\mu(L)$.

For the other direction, we can use the same construction (it is a special case of the the above as $\mathcal{C}_\mu(L) \subseteq \mathcal{C}_0(L)$). Of course, in this case we do not need Claim 3. $\square$

4. Further isomorphisms

4.1. **Localization of non-isomorphic geometries.** In this subsection the language L has just a single 3-ary relation R. We often suppress L in the notation.

In 5.2 of [5] Hrushovski varies his strongly minimal set construction to produce examples where the model-theoretic structure of the strongly minimal set can be read off from the geometry: lines of the geometry have three points, and colinear points correspond to instances of the ternary relation. He thereby produces continuum-many non-isomorphic geometries of (countable, saturated) strongly minimal structures, but asks whether these examples are *locally* isomorphic. We show that this is the case: in fact, localizing any of them over a 4-dimensional set gives a geometry isomorphic to $G(\mathcal{M}_0(L))$, the geometry of the generic structure for $(\mathcal{C}_0(L), \leq)$.

In 5.2 of [5], Hrushovski considers

$$\mathcal{K}_0 = \{A \in \mathcal{C}_0(L) : B \leq A \text{ for all } B \subseteq A \text{ with } |B| \leq 3\}.$$

We state the following without proof:

Lemma 4.1. *With the above notation:*

(i) *If $A \leq B_1, B_2 \in \mathcal{K}_0$ and the free amalgam of B_1 and B_2 over A is not in $\mathcal{K}_0$, then there exist $a, a' \in A$ and $b_i \in B_i \setminus A$ with $R(a, a', b_i)$ holding in B_i (for $i = 1, 2$).*

(ii) *The class $(\mathcal{K}_0, \leq)$ is an amalgamation class.*

More generally, given a function μ as before, we can consider $\mathcal{K}_\mu = \mathcal{K}_0 \cap \mathcal{C}_\mu(L)$ and for appropriate μ, the class $(\mathcal{K}_\mu, \leq)$ will satisfy Assumption 2.2. In fact, we only need to define $\mu(Y, Z)$ for $\delta(Y) \geq 3$. For suppose $Y \leq Z$ is a minimally simply algebraic extension in $\mathcal{K}_0$ and $\delta(Y) = \delta(Z) \leq 2$. Then Z has at most 3 elements: otherwise there is a subset $W \subseteq Z$ of size 3 with $W \notin R^Z$, and then W is not self-sufficient in Z, contradicting the definition of $\mathcal{K}_0$. It follows that the value of $\mu(Y,Z)$ is irrelevant for such $Y \leq Z$: the multiplicity is already controlled by the definition of $\mathcal{K}_0$.

Denote the generic structure of $(\mathcal{K}_\mu, \leq)$ by $\mathcal{N}_\mu$. The d-closure of two points in $\mathcal{N}_\mu$ has size 3 (as above), so certainly $G(\mathcal{N}_\mu)$ and $G(\mathcal{M}_0(L))$ are non-isomorphic. In fact, we can recover the relation R from the geometry $G(\mathcal{N}_\mu)$ as the 3-sets with dimension 2. Thus different μ give different geometries.

We show:

Theorem 4.2. *Suppose $\mu(Y,Z) \geq 3$ for all msa $Y \leq Z$ in $\mathcal{K}_0$ with $\delta(Y) \geq 3$. Let $X \leq \mathcal{N}_\mu$ and $d(X) = 4$. Then the localization $G_X(\mathcal{N}_\mu)$ is isomorphic to $G(\mathcal{M}_0(L))$.*

Proof. Suppose we are given $A \leq B \in \mathcal{C}_0(L)$ and $X \leq A' \in \mathcal{K}_\mu$ with X consisting of 4 points (and no relations) and an isomorphism $f : G(A) \to G_X(A')$. We want to find $B' \in \mathcal{K}_\mu$ with $A' \leq B'$ and an isomorphism $f' : G(B) \to G_X(B')$ extending f. This is very similar to the the construction of B' in the proof of Theorem 3.1 and we will only indicate what needs to be modified and provide extra argument as required.

Let $A_0 = \mathrm{cl}_A(\emptyset)$ and let $A_1, \ldots, A_r$ be the d-dependence classes on $A \setminus A_0$: the latter are the points of $G(A)$. Similarly let $B_0 = \mathrm{cl}_B(\emptyset)$ and $B_1, \ldots, B_s$ the d-dependence classes on $B \setminus B_0$, with $A_i \subseteq B_i$ for $i = 1, \ldots, r$. List the relations on B which are not contained in A or some $B_0 \cup B_j$ as $\rho_1, \ldots, \rho_t$. So these are 3-sets and note that each of them intersects three different B_i. Let $A'_0 = \mathrm{cl}_{A'}(X)$ and $A'_1, \ldots, A'_r$ be the classes of d-dependence over X on $A' \setminus A'_0$, labelled so that $f(A_i) = A'_i$. We construct $B'_0, B'_1, \ldots, B'_s$ with $A'_i \subseteq B'_i$ for $i = 0, \ldots, r$, and $B' = \bigcup_{i=0}^{s} B'_i$ in the following way.

Step 1: Construction of $A'' = A' \cup B'_0 \in \mathcal{K}_\mu$.
This is as before, but we need to take $V \in \mathcal{K}_\mu$: we can do this because algebraic extensions of X can be arbitrarily large.

Step 2: Construction of $B'_0 \cup B'_i \in \mathcal{K}_\mu$.
The construction is as in Theorem 3.1 for $i \leq r$. In the case $i > r$ we vary the construction by taking the s_{ij} to be distinct. The proofs that B'_0 is d-closed in $B'_0 \cup B'_i$ are as before; as are the arguments which show that if $Y \leq Z$ is a msa extension in $\mathcal{K}_0$ with $\delta(Y) \geq 3$ then there are at most $\mu(Y,Z)$ copies of Z over Y in $B'_0 \cup B'_i$. So it remains to show that $B'_0 \cup B'_i \in \mathcal{K}_0$.

If $i \leq r$, then using the amalgmation lemma 4.1 as in Step 2, Case 1 of Theorem 3.1, it will suffice to show that $B'_0 \cup A'_i \cup \{b_{ij}\} \in \mathcal{K}_0$. This is the free amalgam of $\{s_{ij}, b_{i0}, b_{ij}\}$ and $B'_0 \cup A'_i$ over $\{s_{ij}, b_{i0}\}$. So we can apply 4.1 (because $\{s_{ij}, b_{i0}\}$ is in no relation in $B'_0 \cup A'_i$).

Now suppose $i > r$. We analyse the possibilities for $\delta(Y)$ when $Y \subseteq B'_0 \cup B'_i$, $Y \not\subseteq B'_0$ and $|Y| > 1$. As $Y \cap B'_0$ is d-closed in Y we have $\delta(Y) > \delta(Y \cap B'_0)$. If $Y \cap B'_0 = \emptyset$ then by the construction, $\delta(Y) = |Y|$. If $Y \cap B'_0$ is a singleton then Y has at most one relation (because the s_{ij} are distinct), so $\delta(Y) > 1$ and $\delta(Y) \geq |Y| - 1$. In the remaining case, $\delta(Y \cap B'_0) \geq 2$ (as $B'_0 \in \mathcal{K}_0$), so $\delta(Y) \geq 3$. Thus, Y consists of 2 points, or is 3 points in a relation, or has $\delta(Y) \geq 3$. It follows that $B'_0 \cup B'_i \in \mathcal{K}_0$.

Step 3: Other relations on B'.
As before.

Claim 1: Let $U \subseteq \{1, \ldots, s\}$, and $Y = \bigcup_{i \in U}(B_i \cup B_0)$, $Y' = \bigcup_{i \in U}(B'_i \cup B'_0)$. Then $Y \cap A$ is d-closed in A iff $Y' \cap A'$ is d-closed in A', and in this case we have $\delta(Y) = \delta(Y'/B'_0)$.

The same proof works, noting that in B' we work over B'_0.

Claim 2: $B' \in \mathcal{C}_0(L)$, B'_0 and $B'_0 \cup B'_i$ are d-closed in B' and $A'' \leq B'$. The map $f' : G(B) \to G_X(B')$ given by $f'(B_i) = B'_i$ is an isomorphism of geometries which extends f.

This is as before, using the modified version of Claim 1.

Claim 3: If $i \neq j$ then $B'_0 \cup B'_i \cup B'_j \in \mathcal{K}_\mu$ and $B'_0 \cup B'_i \cup B'_j \leq B'$.

By construction $B'_0 \cup B'_i \cup B'_j$ is the free amalgam of $B'_0 \cup B'_i$ and $B'_0 \cup B'_j$ over B'_0, and B'_0 is d-closed in each. So the first statement follows from the assumed amalgamation lemma. The second statement follows from Claim 1: because each $B_0 \cup B_i$ is d-closed in B, the union of at least two of these has δ-value at least 2. $\square_{Claim}$

Claim 4: $B' \in \mathcal{K}_0$.

We need to show that if $D \subseteq B'$ has size at most 3 then $D \leq B'$. If $|D| \leq 2$ then $D \subseteq B'_0 \cup B'_i \cup B'_j$ for some i, j and it follows from Claim 3 that $D \leq B'_0 \cup B'_i \cup B'_j \leq B'$. So suppose D has size 3 and $D \subseteq C$ with $\delta(C) < \delta(D)$. We must have $\delta(C) = 2$ (as any two points of D are self-sufficient in C and have δ-value 2). As $A'' \in \mathcal{K}_0$ there is an i such that $C \cap (B'_i \setminus A'') \neq \emptyset$. Note that C is not contained in $B'_0 \cup B'_i$ (because this is in $\mathcal{K}_\mu$), so as B'_0 is d-closed in $B'_0 \cup B'_i$ and the latter is d-closed in B', we have

$$0 \leq \delta(C \cap B'_0) < \delta(C \cap (B'_0 \cup B'_i)) < \delta(C) = 2.$$

Thus $\delta(C \cap B'_0) = 0$, so $C \cap B'_0 = \emptyset$.

It then follows from Step 2 of the construction that there is no adjacency in C between points of $C \cap A''$ and points of $C \setminus A''$. Let $q = |\{j : \rho'_j \subseteq C\}|$. Then (using $A'' \leq B'$; so $C \cap A'' \leq C$)

$$2 \geq \delta(C/C \cap A'') = |C \setminus A''| - q \geq 2q$$

as the ρ'_j are disjoint. If $q = 0$ then C is $C \cap A''$ together with some isolated points, and this is in $\mathcal{K}_\mu$ (so not possible in this situation). If $q = 1$ then C consists of 3 points in a single relation and this has no subset of the form required for D. $\square_{Claim}$

Claim 5: $B' \in \mathcal{K}_\mu$
We already know that $B' \in \mathcal{K}_0$, so we need to show that $B' \in \mathcal{C}_\mu(L)$, at least as far as msa extensions $Y \leq Z$ with $\delta(Y) \geq 3$ are concerned. So suppose $Z_1, \ldots, Z_4$ are disjoint copies of Z over Y in B'. Then each element $y \in Y$ is in at least 4 relations (using 2.1, as before), so by construction, $y \in A''$. Thus $Y \subseteq A''$ and the rest of the proof is just as in Claim 3 of Theorem 3.1. $\square_{Claim}$

Claims 2 and 5 finish the proof of one direction of the isomorphism extension property. For the other direction, suppose we are given $X \leq A \leq B \in \mathcal{K}_\mu$ and $A' \in \mathcal{C}_0(L)$ with an isomorphism $f : G_X(A) \to G(A')$. We want $A' \leq B' \in \mathcal{C}_0(L)$ and an isomorphism $f' : G_X(B) \to G(B')$ extending f. Let $\bar{X}$ consist of X with the four 3-sets as relations. So $\delta(\bar{X}) = 0$. Let $\bar{A}$ consist of A with X replaced by $\bar{X}$. Define $\bar{B}$ similarly. Then $\bar{A} \leq \bar{B} \in \mathcal{C}_0(L)$ (as in the First Changing Lemma, Lemma 4.1 of [2]). Moreover, as in the Fourth Changing Lemma (Lemma 5.3 of [2]), the map $h : G_X(B) \to G(\bar{B})$ given by $\mathrm{cl}_B(Xb) \mapsto \mathrm{cl}_{\bar{B}}(b)$ is an isomorphism of geometries sending $G_X(A)$ to $G(\bar{A})$. So we have an isomorphism of geometries $\bar{f} : G(\bar{A}) \to G(A')$ (given by $\bar{f} = f \circ (h|A)^{-1}$), and $\bar{A} \leq \bar{B} \in \mathcal{C}_0(L)$, and it will suffice to find $B' \in \mathcal{C}_0(L)$ and $\bar{f}' : G(\bar{B}) \to G(B')$ extending $\bar{f}$. We do this using the construction as at the end of the proof of Theorem 3.1. $\square$

Remark 4.3. The result is also true with $d(X) = 3$: all we really used was that $\mathrm{acl}(X)$ is infinite in the generic structure. The final paragraph of the above proof needs some slight modification in this case.

Remark 4.4. Note that $\mathcal{K}_0$ can be considered as $\mathcal{K}_\mu$ where $\mu(Y, Z)$ is formally given the value ∞ for all msa $Y \leq Z \in \mathcal{K}_0$. Thus the above argument also shows that the geometry of $\mathcal{N}_0$, the generic for $(\mathcal{K}_0, \leq)$, is locally isomorphic to $G(\mathcal{M}_0(L))$.

Remark 4.5. Another variation is given in 5.1 of [5]. Let $k \geq 2$ and consider the language L with a single $(k+1)$-ary relation symbol R.

Let
$$\mathcal{C}_0'(L) = \{A \in \mathcal{C}_0(L) : \delta(B) \geq \min(|B|, k) \ \forall B \subseteq A\}.$$
So if $C \subseteq A \in \mathcal{C}_0'(L)$ and $|C| \leq k$ then $C \leq A$. Hrushovski observes that $(\mathcal{C}_0'(L), \leq)$ is a free amalgamation class and that the assumed amalgamation lemma 2.2 holds for $(\mathcal{C}_\mu', \leq)$, for suitable $\mu \geq 2$. The generic structures here are strongly minimal and any k points are independent. So the geometries are again different from that of $\mathcal{M}_0(L)$. However, they are again locally isomorphic. To see this we proceed as in Theorem 4.2, but take X to be a set of size k. The construction and proof are essentially the same as before, except for in Claim 4 where to show that $B' \in \mathcal{C}_0'(L)$ we modify the argument as follows.

Suppose $C \subseteq B'$ has $\delta(C) < k$ and $|C| \geq k+1$. Then for some i we have:
$$0 \leq \delta(C \cap B_0') < \delta(C \cap (B_0' \cup B_i')) < \delta(C) \leq k-1.$$
So $\delta(C \cap B_0') \leq k-3$ and therefore $|C \cap B_0'| \leq k-3$. Then by construction there is no adjacency in C between points of $C \cap A''$ and points of $C \setminus A''$. So (with q as before):
$$k - 1 \geq \delta(C/C \cap A'') = |C \setminus A''| - q \geq kq.$$
Thus $q = 0$ and we have a contradiction.

4.2. **Changing the language and predimension.** Recall that the language L consists of relation symbols $\{R_i : i \in I\}$ with R_i of arity n_i (and only finitely many symbols of each arity). Suppose that $L_0 = \{R_i : I \in I_0\}$ is a sublanguage with the property that for every $i \in I$ there is $j \in I_0$ such that $n_i \leq n_j$. For example, if I is finite we can take L_0 to consist of a relation symbol of maximal arity in L. The following is essentially Theorem 3.1 of [2], but working with sets rather than tuples: we omit most of the details of the proof.

Theorem 4.6. *The geometries $G(\mathcal{M}_0(L))$ and $G(\mathcal{M}_0(L_0))$ are isomorphic.*

Proof. We can use the construction in Theorem 3.1 to show that $\mathcal{C}_0(L) \rightsquigarrow \mathcal{C}_0(L_0)$ holds. In step 3 of the construction, if ρ_i is a k-set then we take ρ_i' to be a k'-set with $k' \geq k$: the condition on L_0 allows us to do this. Claims 1 and 2 of Theorem 3.1 then go through exactly as previously. The direction $\mathcal{C}_0(L_0) \rightsquigarrow \mathcal{C}_0(L)$ follows as in Theorem 3.1. □

Remark 4.7. Theorem 3.1 of [2] works with a predimension of the form:
$$\delta_\alpha(A) = |A| - \sum_{i \in I} \alpha_i |R_i^A|,$$

where the α_i are natural numbers. We can adapt the argument here to deal with such predimensions. For example, suppose I is finite and R_1 is of maximal arity and $\alpha_1 = 1$. Let L_0 consist of R_1. Then, as in Theorem 3.1 of [2], $G(\mathcal{M}_0(L))$ is isomorphic to $G(\mathcal{M}_0(L_0))$. To show that $\mathcal{C}_0^\alpha(L) \rightsquigarrow \mathcal{C}_0(L_0)$ (where $\mathcal{C}_0^\alpha(L)$ is defined using the predimension δ_α) we perform the same construction except that in step 3, if ρ_j is of type R_i then we add α_i corresponding ρ'_j (but still disjoint etc).

4.3. **Sets versus tuples.** We have chosen to work with structures A where the relations R_i^A are sets of n_i-sets. As was done in [2] we could also have worked more generally with structures A where the R_i^A are sets of n_i-tuples and the predimension is still given by $|A| - \sum_i |R_i^A|$. Let $\hat{\mathcal{C}}_0(L)$ denote the class of these finite structures with $\emptyset \leq A$.

Theorem 4.8. *The geometries of the generic structures of the amalgamation classes $(\mathcal{C}_0, \leq)$ and $(\hat{\mathcal{C}}_0(L), \leq)$ are isomorphic.*

Proof. This is the usual sort of proof using the construction. For example, to show $\hat{\mathcal{C}}_0(L) \rightsquigarrow \mathcal{C}_0(L)$ we replace an n_i-tuple ρ_j (in $R_i^B \setminus R^A$) by an n_i-set, using the new d-dependent points to eliminate repetitions of points in the tuple or different enumerations of the same set. $\square$

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